On an inconsistency in
path integral bosonization

R. Kerschner

Institut für Theoretische Physik, University of Tübingen
Auf der Morgenstelle 14, 72076 Tübingen, Germany

Abstract

A critically discerning discussion of path integral bosonization is given. Successively evaluating the conventional path integral bosonization of QCD it is shown without any approximations that gluons must be composed of two quarks. This contradicts the fundamentals of QCD, where quarks and gluons are independent fields. Furthermore, bosonizing the Fierz reordered effective four quark interaction term yields gluons, too. Colorless "mesons" are shown to be Fierz equivalent to a submanifold of gluons. The results obtained are not specific to QCD, but apply to other models as well.

1 Introduction

Due to constrained dynamical variables the canonical quantization of gauge theories is rather involved. It is commonly believed that a (renormalizable) quantum field theory can equivalently be expressed by a path integral. Restricting ourselves to quantum chromodynamics (QCD), the path integral is formulated in terms of unobservable fields, i. e. quarks and gluons. The observable quantities are considered to be colorless composite particles which are composed of two or three quarks. Thus the path integral quantization of QCD is only complete if it is supplied by a theory of formation and dynamics of composite particles. In the past several efforts were made to develop such effective composite particle theories, e.g. [1], [2]. The general strategy is to eliminate the gluons in favour of an effective quark interaction theory, which is then bosonized in order to obtain phenomenological meson theories. The problem of an exact treatment of the gluon selfinteraction has recently been solved [2] using the field strength approach [3] to QCD. To derive our results we shall make use of this formalism since it admits exact calculations.

Many critical remarks on path integrals have already been given by e. g. Rivers [4]. In addition to these objections it will be shown that no serious composite particle theory can be obtained from path integral bosonization.

2 Gluons as two-quark composites

As our starting point we consider the Euclidean path integral

$$Z = \int D\,[\,\bar{c}\,]\,D\,[\,c\,]\,D\,[\,\bar{q}\,]\,D\,[\,q\,]\,D\,[\,A\,] \exp\left\{ -S_{QCD} - S_{gf} \right\}\,, \tag{1}$$

where

$$S_{QCD} = \int d^4\!x \left\{ \bar{q}\,(\,\gamma_\mu \partial_\mu + m\,)\,q + \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - ig\bar{q}\gamma_\mu t^a q A^a_\mu \right\}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu \tag{2}$$

is the action and field strength tensor of QCD and we have chosen a covariant gauge fixing

$$S_{gf} = \int d^4\!x \left\{ -\frac{1}{2\xi}\left(\partial_\mu A^a_\mu \right)^2 + i\bar{c}_a \left(\Box\delta_{ab} - g f_{abc}\partial_\mu A^c_\mu \right) c_b \right\} \tag{3}$$

with $\bar{c}$ and c being the Faddeev–Popov ghost fields, cf. [5].

Multiplying equation (1) with

$$1 = \int D\,[\,G\,]\,D\,[\,\psi\,]\,\delta\left(G^a_{\mu\nu} - F^a_{\mu\nu}(A) \right)\delta\left(\psi^a - \partial_\mu A^a_\mu \right)$$
$$= \int D\,[\,G\,]\,D\,[\,T\,]\,D\,[\,\psi\,]\,D\,[\,\phi\,] \exp\left\{ \frac{i}{2}\int d^4\!x\, T^a_{\mu\nu}\left(G^a_{\mu\nu} - F^a_{\mu\nu}(A) \right) + \right.$$
$$\left. + i\int d^4\!x\, \phi^a\left(\psi^a - \partial_\mu A^a_\mu \right) \right\} \tag{4}$$

allows us to introduce the field strength tensor as an independent variable. The ψ–field only serves to simplify (3) and the following calculation. Due to the δ–distributions we can replace $F^a_{\mu\nu}(A)$ by $G^a_{\mu\nu}$ and $\partial_\mu A^a_\mu$ by ψ^a in (1). After integrating out the G–field we obtain

$$Z = \int D\,[\,\bar{q}\,]\,D\,[\,q\,]\,D\,[\,T\,]\,D\,[\,\phi\,]\,Z_1\,[\,\bar{q},q,T,\phi\,]\,Z_2\,[\,\bar{q},q,T,\phi\,] \tag{5}$$

with

$$\begin{aligned}
Z_1 &= \int D\,[\,\bar{c}\,]\,D\,[\,c\,]\,D\,[\,\psi\,]\exp\Big\{ -\int d\overset{4}{x}\,\Big[\bar{q}\,(\,\gamma_\mu\partial_\mu + m\,)\,q + \frac{1}{4}T^a_{\mu\nu}T^a_{\mu\nu} + \\
&\qquad\qquad - \frac{1}{2\xi}\psi^a\psi^a - i\phi^a\psi^a + i\bar{c}_a\,(\,\Box\delta_{ab} - gf_{abc}\psi^c\,)\,c_b\Big]\Big\} \tag{6}\\
Z_2 &= \int D\,[\,A\,]\exp\Big\{ -\int d\overset{4}{x}\,\Big[-ig\bar{q}\gamma_\mu t^a q A^a_\mu + i\phi^a\partial_\mu A^a_\mu + iT^a_{\mu\nu}\partial_\mu A^a_\nu + \\
&\qquad\qquad + \frac{i}{2}gT^a_{\mu\nu}f_{abc}A^b_\mu A^c_\nu\Big]\Big\}\,. \tag{7}
\end{aligned}$$

For the further evaluation we can concentrate on (7). Performing partial integrations with respect to the space-time coordinates and integrating out the A–fields gives

$$Z_2 = \Big(\det\tilde{T}\Big)^{-\frac{1}{2}}\exp\Big\{ \frac{1}{2}\int d\overset{4}{x}\,j^a_\mu(\tilde{T}^{-1})^{ab}_{\mu\nu}j^b_\nu\Big\}\,, \tag{8}$$

with j^a_μ and $\tilde{T}^{ab}_{\mu\nu}$ being defined by

$$\begin{aligned}
j^a_\mu &= ig\bar{q}\gamma_\mu t^a q + iR^a_\mu\,(\,\phi,T\,) \\
R^a_\mu\,(\,\phi,T\,) &= \partial_\mu\phi^a + \partial_\nu T^a_{\nu\mu} \\
\tilde{T}^{bc}_{\mu\nu} &= igT^a_{\mu\nu}f_{abc}\,. \tag{9}
\end{aligned}$$

The effective action in (8) contains a four quark interaction term, coupled to the (inverse) field strength tensor. It is generally argued, that this interaction can be replaced by a contact force at low energy, yielding a Nambu–Jona-Lasinio type of model. These models have extensively been studied for the purpose of deriving effective mesonic actions in the low energy regime, e.g. [6]. Hence hopefully (8) and (5) will give a microscopic derivation of effective meson theories from QCD, with the mesons being $\bar{q}q$–composites. Depending on the energy scale we expect to observe form factors, exchange force corrections stemming from the quark level, and excited meson states.

Obviously there are several (equivalent) possibilities to select quark pairings for the bosonization of (8). The most elucidatory way is to directly choose $g\bar{q}\gamma_\mu t^a q \to \xi^a_\mu$, although the composites are no color singlets in that case. Since one knows that observable mesons must be color singlets this choice is not standard. Rather one usually applies a Fierz reordering to (8) before bosonizing it. However, the Fierz transformation is a strict identity on the quark level and

we will show that it possesses an exact counterpart on the bosonized level. On account of their coloredness we shall call the above composites pseudo-mesons and postpone the discussion of the effects of a Fierz reordering to the following section.

To simplify notation we combine Lorentz, color, and flavor degrees of freedom to a single index

$$
(\gamma_\mu)_{l_1 l_2} (t^a)_{c_1 c_2} (\mathbb{1})_{f_1 f_2} = \Lambda^A_{i_1 i_2} .
\tag{10}
$$

Equation (8) is then bosonized[1] by multiplying it with

$$
\begin{aligned}
1 &= \int D[\xi] \delta \left(\xi^A - g\bar{q}\Lambda^A q \right) \\
&= \int D[\xi] D[\eta] \exp \left\{ \int d^4x \, i\eta^A \left(\xi^A - g\bar{q}\Lambda^A q \right) \right\}
\end{aligned}
\tag{11}
$$

and replacing $g\,\bar{q}\Lambda^A q$ by ξ^A, which gives

$$
\begin{aligned}
Z_2 &= \int D[\xi] D[\eta] \left(\det \tilde{T} \right)^{-\frac{1}{2}} \exp \left\{ \int d^4x \left[i\eta^A \left(\xi^A - g\bar{q}\Lambda^A q \right) + \right. \right. \\
&\quad \left. \left. -\frac{1}{2} \left(\xi^A + R^A \right) (\tilde{T}^{-1})^{AB} \left(\xi^B + R^B \right) \right] \right\} .
\end{aligned}
\tag{12}
$$

Herein the η–coordinates describe the pseudo-mesons, while the ξ–coordinates have to be eliminated. Integrating out the ξ–coordinates and making use of (10) and (9) yields

$$
\begin{aligned}
Z_2 &= \int D[\eta] \exp \left\{ \int d^4x \left[-\frac{i}{2} g T^a_{\mu\nu} f_{abc} \eta^b_\mu \eta^c_\nu + \right. \right. \\
&\quad \left. \left. + i\eta^a_\mu \partial_\mu \phi^a + i\eta^a_\mu \partial_\nu T^a_{\nu\mu} - ig\bar{q}\gamma_\mu t^a q \eta^a_\mu \right] \right\} ,
\end{aligned}
\tag{13}
$$

since the determinant from the ξ–integration exactly compensates the determinant from the A–integration. Equation (13) coincides exactly with (7) after partial integration and renaming of $\eta^a_\mu \to A^a_\mu$. Inserting (13) into (5) we can integrate out T, ϕ and ψ to get (1).

Hence, the pseudo-mesons are exactly the gluons (fixed in the same gauge). Thus, if composite particle actions are obtained from path integral bosonization, then the gluons must be composite particles, too! This exceeds by far pair creation and annihilation phenomena, since there should be form factor effects and fermionic exchange forces in gluon-gluon scattering. In addition, we should have excited gluon states. As a matter of principle this result contradicts the original assumptions of QCD, that gluons are described as independent fields.

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[1]This bosonization technique is fully equivalent to the one used by other authors [1], where one multiplies the path integral with a Gauss integral and subsequently applies a shift in order to eliminate the four quark interaction term.

3 The effect of a Fierz reordering

Applying a Fierz transformation to the four quark interaction term in (8) yields (among others) a color singlet channel of two-quark pairings. Bosonizing the resulting channels will therefore contain bosonic coordinates with the quantum numbers of phenomenological mesons. Hence one might believe that these coordinates will describe mesons. Subsequently we will show that this is not the case.

Extending the Λ–matrices to a complete and orthonormalized set of matrices M^r

$$tr\left(M^r M^{r'}\right) = \delta_{rr'}\,, \qquad \sum_r M^r_{i_1 i_2} M^r_{i_3 i_4} \;=\; \delta_{i_1 i_4}\delta_{i_2 i_3} \tag{14}$$

the Fierz transformation of (8) reads

$$Z_2 \;=\; \left(\det\tilde{T}\right)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}g^2\bar{q}M^r q(\tilde{T}^{-1})^{AB}m_{rs}^{AB}\bar{q}M^s q +\right.$$
$$\left. -gR^A(\tilde{T}^{-1})^{AB}\bar{q}\Lambda^B q - \frac{1}{2}R^A(\tilde{T}^{-1})^{AB}R^B\right\}, \tag{15}$$

where the Fierz matrix is given by[2]

$$m_{rs}^{AB} = -tr\left(\Lambda^A M^r \Lambda^B M^s\right)\,. \tag{16}$$

Bosonizing (15) by multiplication with

$$1 \;=\; \int D\left[\xi\right]\delta\left(\xi^r - g\bar{q}M^r q\right)$$
$$=\; \int D\left[\xi\right]D\left[\eta\right]\exp\left\{\int d\overset{4}{x}\, i\eta^r\left(\xi^r - g\bar{q}M^r q\right)\right\} \tag{17}$$

yields

$$Z_2 \;=\; \int D\left[\xi\right]D\left[\eta\right]\left(\det\tilde{T}\right)^{-\frac{1}{2}}\exp\left\{\int d\overset{4}{x}\left[i\eta^r\xi^r - \frac{1}{2}\xi^r(\tilde{T}^{-1})^{AB}m_{rs}^{AB}\xi^s +\right.\right.$$
$$\left.\left. -ig\bar{q}M^r q\eta^r - R^A(\tilde{T}^{-1})^{AB}\xi^B - \frac{1}{2}R^A(\tilde{T}^{-1})^{AB}R^B\right]\right\}, \tag{18}$$

but contrary to (12) the ξ–integration is now more involved. This is due to the fact that the Fierz matrix satisfies the projector equation

$$m_{rs}^{AB} = m_{r's'}^{AB}\, P_{rs}^{r's'}\,, \tag{19}$$

where the projector $P_{rs}^{r's'}$ is given by

$$P_{rs}^{r's'} = m_{rs}^{AB}\, m_{r's'}^{AB}\,. \tag{20}$$

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[2]One might also consider other Fierz transformations, but none change the following line of arguments.

Therefore the kernel $(\tilde{T}^{-1})^{AB} m_{rs}^{AB}$ has eigenvalues zero, effectively restricting the non-trivial domain of the η–integration.

Nevertheless the ξ–integration can be performed exactly by taking into consideration that the ξ–coordinates are not ordinary boson coordinates, but bosonized Grassmann variables, which must be integrated over by corresponding Berezin rules. These rules are derived from (17) and read[3]

$$\int D[\xi] \, F(\xi^r) = \int D\mu[\bar{\alpha}, \alpha] \, F(\bar{\alpha} M^r \alpha) \,, \tag{21}$$

where $\bar{\alpha}$ and α are anticommuting Grassmann variables and the measure is

$$D\mu[\bar{\alpha}, \alpha] = D[\bar{\alpha}] \, D[\alpha] \, \mu(\bar{\alpha} M^r \alpha) \tag{22}$$

with

$$\mu(\bar{\alpha} M^r \alpha) = \left[\int D[\eta] \, \det(-i\eta^r M^r) \exp\left\{ \int d^4\!x \, i\eta^r \bar{\alpha} M^r \alpha \right\} \right]^{-1}, \tag{23}$$

and F denotes an arbitrary functional. Hence the bosonization forces us to integrate out the ξ–coordinates by the bosonized Berezin rules (21).

Applying (21) to equation (18) yields

$$Z_2 = \int D[\xi] \, D[\eta] \left(\det \tilde{T} \right)^{-\frac{1}{2}} \exp\left\{ \int d^4\!x \Big[i\xi^r \eta^r - ig\bar{q} M^r q\eta^r + \right.$$
$$\left. -\frac{1}{2} \left(\xi^A + R^A \right) (\tilde{T}^{-1})^{AB} \left(\xi^B + R^B \right) \Big] \right\}, \tag{24}$$

if we make use of the bosonized Fierz identity

$$\int D[\xi] \, F\left(-\tfrac{1}{2} \xi^r (\tilde{T}^{-1})^{AB} m_{rs}^{AB} \xi^s \right) = \int D[\xi] \, F\left(-\tfrac{1}{2} \xi^A (\tilde{T}^{-1})^{AB} \xi^B \right). \tag{25}$$

This identity is a consequence of (21) alone and allows us to remove the eigenvalues zero in the kernel $(\tilde{T}^{-1})^{AB} m_{rs}^{AB}$ from (18). Since (25) is an identity, the zero eigenspace of the projector (20) is actually not in the domain of the ξ–integration, if this integration is evaluated by the bosonized Berezin rules (21).

The determinant in (24) can be expressed by a boson path integral, which gives

$$Z_2 = \int D[A] \, D[\xi] \, D[\eta] \exp\left\{ \int d^4\!x \Big[i\xi^r \eta^r - i\xi^B A^B + \right.$$
$$\left. -\frac{1}{2} \tilde{T}^{BC} A^B A^C - iR^B A^B - ig\bar{q} M^r q\eta^r \Big] \right\}. \tag{26}$$

If the Fourier decomposition of the δ–function in (17) is to be consistent with exchanging the order of ξ– and η–integration, we must have

$$\delta(\chi^r) = \int D[\xi] \exp\left\{ \int d^4\!x \, i\xi^r \chi^r \right\}. \tag{27}$$

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[3]In order to prove this formula, one must simply bosonize the right hand side of (21) by multiplication with (17).

Integrating out the ξ–coordinates in (26), we therefore get

$$
\begin{aligned}
Z_2 \;=\; & \int D\,[\,A\,]\, D\,[\,\eta\,]\,\delta\left(\eta^r - \delta^r_B A^B\right)\times \\
& \times \exp\left\{-\frac{1}{2}\tilde{T}^{BC}A^B A^C - iR^B A^B - ig\bar{q}M^r q\eta^r\right]\right\}. \qquad (28)
\end{aligned}
$$

Either one of the A– or η–integration yields (13) again.

Thus a Fierz reordering does not give any new results. Rather the Fierz transformed integral (18) conceals the non-trivial domain of the η–integration, if the ξ–coordinates are not integrated over by bosonized Berezin rules. Any "approximation" of solely selecting specific channels and forgetting about the Grassmann properties of the ξ–integration will definitely change the theory in a way which is no longer consistent with QCD.

Taking the steps (28) to (18) backwards, it becomes even more obvious that the η–coordinates in (18) describe gluons rather than mesons or pseudo-mesons. In (28) the η–coordinates are then directly introduced as new names for the gluons. Note that we do not need any kind of $\bar{q}q$–clustering, if (18) is derived in this way. In particular we would have obtained equation (18) without the η–quark coupling term from the pure Yang–Mills action $S_{YM} = \int d^4x\,\frac{1}{4}\,F^a_{\mu\nu}F^a_{\mu\nu}$ instead of S_{QCD} in (1).

4   Conclusions

The crucial point of our results is the triviality of the bosonization procedure.

If path integral bosonization is taken to be a meaningful tool for deriving effective composite particle actions then, in QCD, we are *forced* to regard the gluons as $\bar{q}q$–composites. This contradicts the fundamentals of QCD, since gluons and quarks are introduced as elementary and independent fields there, but by path integral bosonization the gluons are turned into dependent composite fields.

We must therefore reject path integral bosonization as a technique to obtain serious composite particle theories. The simple boson-like character of coordinates (possibly associated with correct total quantum numbers) in the path integral does not exhaust the rich structure of a true composite particle theory in that case and cannot guarantee for the derivation of mesons as observable states.

Further we have shown that "mesons" obtained by a Fierz reordering and carrying different quantum numbers as the gluons are restricted variables in the path integral. If the path integral is evaluated correctly by bosonized Berezin rules, the "mesons" turn out to be gluons again. This shows that colorlessness is not a sufficient condition for detecting mesons.

Note finally that no use was made of special properties of QCD. Therefore our objections on path integral bosonization apply to other models as well.

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